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BOUNDARY LAYERS OF INVISCID COMPRESSIBLE NON-ISENTROPIC FLOW IN HALF SPACE

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1. INTRODUCTION

In this paper, we consider the asymptotic behavior of inviscid compressible non-isentropic flow in half space of $\mathbb{R}^d, d=2,3$, as the heat-conductivity tends to zero. As Prandtl found in his pioneering work [30], away from the boundary the flow is mainly driven by the inertia while the friction force is negligible, where the flow can be described approximately as inviscid, and the friction force plays a key role in determining the behavior of flow near physical boundaries, in which it generates large vortices. On the other hand, as observed by many physicians and mechanicians, cf. [31, 32, 28, 2, 17, 37], the heat conduction also has large gradient near physical boundaries, thus it is challenging and very important to analyse the flow behavior near boundaries. In our case, the boundary layers can be describe by the following initial-boundary value problem in $\{(t,x',y): \ t>0, x'\in\mathbb{R}^{d-1}, y\in\mathbb{R}_+\}$:

$$
\begin{cases}
\partial_t\mathbf{u}_h+(\mathbf{u}_h\cdot\nabla_h+u_d\partial_y)\mathbf{u}_h=0,\\
\partial_t\theta+(\mathbf{u}_h\cdot\nabla_h+u_d\partial_y)\theta=\frac{\kappa}{P}\theta\partial_y^2\theta+\frac{\kappa P_t}{P}\theta,\\
\nabla_h\cdot\mathbf{u}_h+\partial_y u_d=\frac{\kappa}{P}\partial_y^2\theta-\frac{(1-\kappa)P_t}{P},\\
(u_d,\theta)|_{y=0}\big(0,\theta^0(t,x')\big), \qquad \lim\limits_{y\to+\infty}\theta(t,x,y)=\Theta(t,x'),\\
(\mathbf{u}_h,\theta)|_{t=0}=(\mathbf{u}_{h0},\theta_0)(x',y),
\end{cases}
\tag{1.1}
$$

where $P=P(t)$ and $\Theta(t,x')$ are positive known functions, and $\kappa>0$ is a constant. One can find the derivation of (1.1) in Appendix. This paper is devote to show the local existence of smooth solution of boundary layers problem (1.1) without monotonicity condition for initial data, and give a simple solution which forms a singularity in finite time. The limit of smooth solutions is studied when the temperature tends to a constant state. Then, we consider the linear stability of the problem (1.1) around a shear flow.

The mathematical theory and stability analysis of boundary layers in fluids are classical and challenging problems. The concept of boundary layers was introduced in Prandtl's seminal work [30] in 1904, and its importance in mechanics was first studied by Prandtl in [31, 32]. It was also studied throughly by many mechanicians in [2, 17, 28, 35, 37] and references therein. Oleinik and her collaborators in [29], obtained the well-posedness of the two-dimensional Prandtl boundary layer equations in viscous incompressible flow in the class of tangential velocity being strictly monotonic with respect to the normal variable by introducing the von Miss or Crocco transformation. Recently, Oleinik's well-posedness result was reproved by the energy method without using the Crocco transformation by [1] and [27] independently. Under the Oleinik monotonicity assumption and an additional condition of favorite pressure, Xin and Zhang ([43]) obtained a global weak solution to the two-dimensional Prandtl equations. Without the monotonicity assumption of the velocity field, the Prandtl equations are ill-posedness in general in the finite order Sobolev spaces of solutions, it is related with the separation of boundary layers, it is studied by many mathematicians, cf. [3, 4, 5, 6, 8, 10, 12]. The well-posedness of the Prandtl equations in the frame of analytic solutions was studied in [16, 23, 33], and the almost global existence of analytic solutions was obtained recently by [13, 44]. Recently, the authors [20, 21, 22] have studied the stability and instability of the Prandtl equations in three space variables. Among the mathematical problems of boundary layers, the convergence from the Navier-Stokes equations to the composition of the Euler equations for outflow and equations of boundary layers in the small viscosity limit is almost completely un-known, except some special cases, such as under additional diffusion conditions of energy [14, 15, 38], for circularly symmetric flows [24] or anisotropic viscous flows [18], in the space of analytic solutions [34], under the assumption of support of vortices being away from boundary [25], a steady flow over a moving plate [11], the problems of non-characteristic boundary layers [9, 26], the flows with Navier-slip boundary conditions [41] and so on. All above theoretical works were focused on the incompressible flow, for which there are only viscous boundary layers near the boundaries. As studied in [2, 17, 35, 37], the compressible flow near boundary is much complicated, there are not only viscous layers but also thermal layers, in which the heat transfers quickly, and there exists interaction between viscous layers and thermal layers. There are a few works on compressible viscous flows, the linearized compressible Prandtl equations were studied in [42], the well-posedness of the Prandtl equations in two-dimensional isentropic compressible flows in the monotonic class of tangential velocity was studied in [39] and [7] independently, and the large Mach number limit of the compressible Prandtl equations was also considered in [7], the small viscosity limit of the compressible isentropic viscous flow with the Navier-slip condition was obtained in [40], and the Kato theory of the limit from the Navier-Stokes equations to the Euler equations was extended to the compressible flow in [36]. These results are only limited to the isentropic flow which avoids the thermal layers, and so far there are few mathematical theories on the thermal layers. In [19], we have studied the behavior of viscous layers and thermal layers in nonisentropic compressible circularly symmetric flows with the viscosity and heat conductivity having the same scale.

2. Main Results

Firstly, we show the local existence of solutions to problem (1.1). Let I_k be the $k\times k$ identity matrix for some integer k, and denote $det(A)$ by the determinant of the matrix A. Then, we improve the method of [12] and have the following local existence result for the above problem (1.1), where no monotonicity condition is required on the initial data.

Proposition 2.1. *Let $u_{h0}(x',y),\theta_0(x',y),\theta^0(t,x'),P(t)$ and $\Theta(t,x')$ be smooth functions satisfying that*

$$t^* := \inf\big\{t:\ \inf_{(x',y)\in\mathbb{R}^d_+} det(I_{d-1}+s\nabla_h u_{h0}(x',y))>0,\ \forall s\in[0,t]\big\}>0. \tag{2.1}$$

and there exists a positive constant C_0 such that for $t\in[0,t^*)$ and $(x',y)\in\mathbb{R}^{d-1}_+$,

$$\|u_{h0}\|_{C^2},\ \|\theta_0\|_{C^1},\ |P_t|\le C_0,\quad C_0^{-1}\le \theta_0(x',y),\ \theta^0(t,x'),\ \Theta(t,x'),\ P(t)\le C_0. \tag{2.2}$$

Then, there exists a $t_0: 0<t_0\le t^*$ and a unique classical solution to (1.1) in $[0,t_0)\times\mathbb{R}^d_+$ given by

$$\begin{aligned}
u_h(t,x',y) =& u_{h0}\Big(\xi(t,x',\eta(t,x',y)),\eta(t,x',y)\Big),\quad \theta(t,x',y)=\tilde\theta(t,x',\eta(t,x',y)),\\
u_d(t,x',y) =& \int_0^{\eta(t,x',y)}\partial_t(\frac{\tilde\theta}{a})(t,x',z)dz+\int_0^{\eta(t,x',y)}\Big[b(t,x',\eta(t,x',y))\cdot\nabla_h(\frac{\tilde\theta}{a})(t,x',z)\Big]dz.
\end{aligned} \tag{2.3}$$

Here, the vector function $\xi(t,x',z)\in\mathbb{R}^{d-1}$ is the solution of

$$x' \ =\ \xi+tu_{h0}(\xi,z); \tag{2.4}$$

the scale function $a(t,x',z)$ and vector function $b(t,x',z)$ are given by the initial data and $\xi(t,x',z)$:

$$\begin{cases} a(t,x',z) \ :=\ \frac{P(t)}{P(0)}\theta_0(\xi(t,x',z),z)\cdot det(I_{d-1}+t\nabla_h u_{h0})(\xi(t,x',z),z),\\ b(t,x',z)\ :=\ u_{h0}(\xi(t,x',z),z); \end{cases} \tag{2.5}$$

the function $\tilde\theta(t,x',z)$ is a positive smooth solution to the following problem in $\{0<t\le t_0, z>0, x'\in\mathbb{R}^{d-1}\}$,

$$\begin{cases} \partial_t\tilde\theta+b\cdot\nabla_h\tilde\theta-\frac{\kappa P_t}{P}\tilde\theta-\frac{\kappa a}{P}\partial_z(\frac{a}{\tilde\theta}\partial_z\tilde\theta)=0,\\ \tilde\theta|_{z=0}=\theta^0(t,x'),\quad \lim\limits_{z\to+\infty}\tilde\theta=\Theta(t,x'),\\ \tilde\theta|_{t=0}=\theta_0(x',z); \end{cases} \tag{2.6}$$

and then the scale function $\eta(t,x',y)$ is the solution of

$$y\ =\ \int_0^\eta \frac{\tilde\theta(t,x',z)}{a(t,x',z)}dz. \tag{2.7}$$

Proof. We use the method of characteristics to solve the problem (1.1). We assume that (u, v, θ) is a smooth solution to (1.1), and introduce characteristic coordinates:

$$t = \tau,\ x' = x'(\tau, \xi, \eta),\ y = y(\tau, \xi, \eta)$$

with $\xi = (\xi_1, \cdots, \xi_{d-1}) \in \mathbb{R}^{d-1}$, which satisfies that

$$\begin{cases} \partial_\tau(x', y)(\tau, \xi, \eta) &= (\mathbf{u}_h, u_d)\big(\tau, x'(\tau, \xi, \eta), y(\tau, \xi, \eta)\big), \\ (x', y)(0, \xi, \eta) &= (\xi, \eta). \end{cases} \tag{2.8}$$

Then, let

$$(\bar{\mathbf{u}}_h, \bar{u}_d, \bar{\theta})(\tau, \xi, \eta) := (\mathbf{u}_h, u_d, \theta)\big(\tau, x'(\tau, \xi, \eta), y(\tau, \xi, \eta)\big), \tag{2.9}$$

the equations of (1.1) are deduced as

$$\begin{cases} \partial_\tau \bar{\mathbf{u}}_h &= 0, \\ \partial_\tau \bar{\theta} &= \frac{\kappa}{P}\bar{\theta}\partial_y^2\bar{\theta} + \frac{\kappa P_\tau}{P}\bar{\theta}, \end{cases} \tag{2.10}$$

with the initial data:

$$(\bar{\mathbf{u}}_h, \bar{\theta})|_{\tau=0} = (\mathbf{u}_{h0}, \theta_0)(\xi, \eta). \tag{2.11}$$

Combining (2.10) with (2.11), it is easy to obtain

$$\bar{\mathbf{u}}_h(\tau, \xi, \eta) \equiv \mathbf{u}_{h0}(\xi, \eta), \tag{2.12}$$

Plugging (2.12) into (2.8) yields

$$x' = \xi + \tau \mathbf{u}_{h0}(\xi, \eta). \tag{2.13}$$

Note that from (2.13), $\nabla_\xi x' = I_{d-1} + \tau\nabla_\xi \mathbf{u}_{h0}$ with the derivative operator $\nabla_\xi = (\partial_{\xi_1}, \cdots, \partial_{\xi_{d-1}})^T$, which is positive for $0 \le \tau < t^*$ by virtue of (2.1), it implies that the equation (2.13) is invertible to give $\xi = \xi(\tau, x', \eta)$ when $0 \le \tau < t^*$.

Next, denote by $J(\tau, \xi, \eta)$ the Jacobian of the transformation between (x', y) and (ξ, η):

$$J(\tau, \xi, \eta) := \frac{\partial(x', y)}{\partial(\xi, \eta)} = \det(\nabla_\xi x') \cdot \partial_\eta y - \sum_{i=1}^{d-1}\big[det(\nabla_i x') \cdot \partial_{\xi_i} y\big],$$

where the derivative operators

$$\nabla_i = (\cdots, \partial_{\xi_{i-1}}, \partial_\eta, \partial_{\xi_{i+1}}, \cdots)^T,\ 1 \le i \le d-1.$$

From (2.8), we have $J(0, \xi, \eta) = 1$, and by combining with the divergence condition in (1.1),

$$\begin{aligned} \partial_\tau J(\tau, \xi, \eta) &= J(\tau, \xi, \eta) \cdot (\nabla_h \cdot \mathbf{u}_h + \partial_y u_d)\big(\tau, x'(\tau, \xi', \eta), y(\tau, \xi', \eta)\big) \\ &= J(\tau, \xi, \eta) \cdot \Big[\frac{\kappa}{P(\tau)}\partial_y^2\bar{\theta}(\tau, \xi, \eta) - (1-\kappa)\frac{P_\tau(\tau)}{P(\tau)}\Big], \end{aligned}$$

which implies that by virtue of (2.10) and (2.11),

$$J(\tau, \xi, \eta) = \frac{P(0)}{P(\tau)\theta_0(\xi, \eta)}\bar{\theta}(\tau, \xi, \eta). \tag{2.14}$$

From (2.13), we have

$$\det(\nabla_\xi x')(\tau, \xi, \eta) = det\Big(I_{d-1} + \tau\nabla_\xi \mathbf{u}_{h0}(\tau, \xi, \eta)\Big) > 0, \quad \text{for } \tau \le t^*,$$

then combining with the above relation (2.14), it follows that

$$\partial_\eta y - \sum_{i=1}^{d-1}\Big[\frac{det(\nabla_i x')}{det(\nabla_\xi x')} \cdot \partial_{\xi_i} y\Big] = \frac{P(0)}{P(\tau)\theta_0(\xi, \eta) \cdot det(\nabla_\xi x')(\tau, \xi, \eta)}\bar{\theta}(\tau, \xi, \eta). \tag{2.15}$$

We obtain that by calculation, the characteristics of the equation (2.15) are $x' = constant$ or $\xi = \xi(\tau, x', \eta)$ given in (2.13), then denote by

$$\tilde{\theta}(\tau, x', \eta) \ := \ \bar{\theta}\big(\tau, \xi(\tau, x', \eta), \eta\big), \tag{2.16}$$

and it implies that from (2.15),

$$\begin{aligned} \frac{\partial}{\partial_\eta} y(\tau, \xi(\tau, x', \eta), \eta) \ &= \ \frac{P(0)}{\theta_0(\xi(\tau, x', \eta), \eta) \cdot det(\nabla_\xi x')(\tau, \xi(\tau, x', \eta), \eta)} \bar{\theta}(\tau, \xi(\tau, x', \eta), \eta) \\ &= \ \frac{\tilde{\theta}(\tau, x', \eta)}{a(\tau, x', \eta)}, \end{aligned} \tag{2.17}$$

where the scale function $a(\tau, x', \eta)$ is given in (2.5). Moreover, as $u_d|_{y=0} = 0$ and the characteristic equation (2.8), the boundary $y = 0$ is a particle path, thus we may set $y = 0$ when $\eta = 0$. Therefore, integrating (2.17) along characteristics, we obtain

$$y \ = \ y(\tau, \xi(\tau, x', \eta), \eta) \ = \ \int_0^\eta \frac{\tilde{\theta}(\tau, x', z)}{a(\tau, x', z)} dz. \tag{2.18}$$

Consequently, when $0 \leq \tau < t^*$ and $\tilde{\theta} > 0$, we have that $a > 0$ from the definition (2.5), and then by virtue of (2.18),

$$\partial_\eta y = \frac{\tilde{\theta}(\tau, x', \eta)}{a(\tau, x', \eta)} > 0,$$

the equation (2.18) is invertible to give $\eta = \eta(\tau, x', y)$ with

$$\eta_y = \frac{a(\tau, x', \eta)}{\tilde{\theta}(\tau, x', \eta)} > 0. \tag{2.19}$$

Moreover, the domain $\{y > 0\}$ is changed as $\{\eta > 0\}$ with the boundary $\{y = 0\}$, $y \to +\infty$ respectively, being changed as $\{\eta = 0\}$, $\eta \to +\infty$ respectively.

Now, we will derive the formula (2.3) and the problem (2.16) for $\tilde{\theta}(\tau, x', \eta)$. Note that the inverse function of $(x', y)(\tau, \xi, \eta)$ is

$$\Big(\xi(\tau, x', \eta(\tau, x', y)), \eta(\tau, x', y)\Big)$$

given by (2.13) and (2.18). Thus, combining (2.9), (2.12) and (2.16) yields that

$$\mathbf{u}_h(\tau, x', y) = \mathbf{u}_{h0}\Big(\xi(\tau, x', \eta(\tau, x', y)), \eta(\tau, x', y)\Big), \quad \theta(\tau, x', y) = \tilde{\theta}\big(\tau, x', \eta(\tau, x', y)\big),$$

which gives the formulas of $\mathbf{u}_h(t, x', y)$ and $\theta(t, x', y)$ in (2.3). Denote by

$$\tilde{y}(\tau, x', \eta) \ := \ \int_0^\eta \frac{\tilde{\theta}(\tau, x', z)}{a(\tau, x', z)} dz,$$

then from (2.18) and (2.13) we have $y(\tau, \xi, \eta) = \tilde{y}\big(\tau, \xi + \tau \mathbf{u}_{h0}(\xi, \eta), \eta\big)$, which yields that

$$y_\tau(\tau, \xi, \eta) = \partial_\tau \tilde{y}\big(\tau, \xi + \tau \mathbf{u}_{h0}(\xi, \eta), \eta\big) + \mathbf{u}_{h0}(\xi, \eta) \cdot \nabla_h \tilde{y}\big(\tau, \xi + \tau \mathbf{u}_{h0}(\xi, \eta), \eta\big). \tag{2.20}$$

Combining (2.8) with (2.20), we get that

$$\begin{aligned} &u_d\big(\tau, x'(\tau, \xi, \eta), y(\tau, \xi, \eta)\big) = y_\tau(\tau, \xi, \eta) \\ &= \int_0^\eta \partial_\tau(\frac{\tilde{\theta}}{a})(\tau, \xi + \tau \mathbf{u}_{h0}(\xi, \eta), z) dz + \int_0^\eta \mathbf{u}_{h0}(\xi, \eta) \cdot \nabla_h(\frac{\tilde{\theta}}{a})(\tau, \xi + \tau \mathbf{u}_{h0}(\xi, \eta), z) dz, \end{aligned}$$

which implies the formula of $u_d(t, x, y)$ in (2.3) by using that (2.13) and (2.18). Next, from (2.16) and the relation (2.13) we have

$$\bar{\theta}(\tau, \xi, \eta) \ = \ \tilde{\theta}\Big(\tau, \xi + \tau \mathbf{u}_{h0}(\xi, \eta), \eta\Big),$$

and then,

$$\partial_\tau \bar{\theta} = \partial_\tau \tilde{\theta} + \mathbf{u}_{h0} \cdot \nabla \tilde{\theta}, \qquad \nabla_\xi \bar{\theta} = \left(I_{d-1} + \tau \nabla_\xi \mathbf{u}_{h0}\right) \cdot \nabla_h \tilde{\theta},$$
$$\partial_\eta \bar{\theta} = \partial_\eta \tilde{\theta} + \tau \partial_\eta \mathbf{u}_{h0} \cdot \nabla_h \tilde{\theta}.$$

Moreover, from (2.13) it follows that

$$\left(I_{d-1} + \tau \nabla_\xi \mathbf{u}_{h0}\right) \cdot \xi_\eta + \tau \partial_\eta \mathbf{u}_{h0} = 0,$$

thus we obtain that by virtue of (2.19),

$$\partial_y \bar{\theta}(\tau, \xi(\tau, x', \eta), \eta) = \left[(\xi_\eta \cdot \nabla_\xi + \partial_\eta)\bar{\theta}\right](\tau, \xi(\tau, x', \eta), \eta) \cdot \eta_y = \partial_\eta \tilde{\theta}(\tau, x', \eta) \cdot \frac{a(\tau, x', \eta)}{\tilde{\theta}(\tau, x', \eta)}.$$

Therefore, the problem for $\bar{\theta}$ given in (2.10) and (2.11) can be reduced as follows,

$$\begin{cases} \partial_\tau \tilde{\theta} + \mathbf{u}_{h0}\big(\xi(\tau, x', \eta), \eta\big) \cdot \nabla_h \tilde{\theta} = \frac{\kappa a(\tau, x', \eta)}{P(\tau)} \partial_\eta \Big(\frac{a(\tau, x', \eta)}{\tilde{\theta}} \partial_\eta \tilde{\theta}\Big) + \frac{\kappa P_\tau(\tau)}{P(\tau)} \tilde{\theta}, & \text{in } [0, t^*) \times \mathbb{R}^d_+, \\ \tilde{\theta}|_{\tau=0} = \theta_0(x', \eta). \end{cases}$$

Then, from the boundary conditions of θ in (1.1), we get

$$\tilde{\theta}|_{\eta=0} = \theta^0(\tau, x'), \qquad \lim_{\eta \to +\infty} \tilde{\theta}(\tau, x', \eta) = \Theta(\tau, x'),$$

so we obtain the problem (2.6) for $\tilde{\theta}(\tau, x', \eta)$. Then, by the standard Picard method, we may know that the problem (2.6) admits a positive classical solution in $[0, t_0) \times \mathbb{R}^d_+$ for some $t_0 \le t^*$. Finally, One can check directly that (2.3)-(2.7) defines a smooth solution to the problem (1.1).

□

Remark 2.1. *From* (2.3) *and* (2.7) *with the definition of the function a in* (2.5), *one can deduce that there may be a loss of derivatives in the horizontal variables x' for the solution to* (1.1).

2.1. **Singularity formation.** In this subsection, we establish a singular solution of problem (1.1) based on the inviscid Prandtl equations. For this, we consider a simple case of the problem (1.1) with the constant outflow, i.e., functions $P(t)$ and $\Theta(t, x')$ are constants. More precisely, we consider the following problem

$$\begin{cases} \partial_t \mathbf{u}_h + (\mathbf{u}_h \cdot \nabla_h + u_d \partial_y)\mathbf{u}_h = 0, \\ \partial_t \theta + (\mathbf{u}_h \cdot \nabla_h + u_d \partial_y)\theta = \theta \partial_y^2 \theta, \\ \nabla_h \cdot \mathbf{u}_h + \partial_y u_d = \partial_y^2 \theta, \\ (u_d, \theta)|_{y=0} = \big(0, \theta^0(t, x')\big), \quad \lim_{y \to +\infty} \theta(t, x, y) = 1, \\ (\mathbf{u}_h, \theta)|_{t=0} = (\mathbf{u}_{h0}, \theta_0)(x', y). \end{cases} \tag{2.21}$$

Then, the following proposition shows the singularity formation of the solution to (2.21).

Proposition 2.2. *Assume that the initial-boundary data of problem* (2.21) *is given by*

$$(\boldsymbol{u}_{h0}, \theta_0)(x', y) \;=\; \Big(\boldsymbol{U}(y + f_0(x')), 1\Big), \quad \theta^0(t, x') = 1,$$

where $\boldsymbol{U}(y) = \big(U_1(y), U_2(y)\big)$ and $f_0(x')$ are smooth functions. Then, there exists a solution to (2.21) *with $\theta(t, x', y) \equiv 1$ and*

$$\boldsymbol{u}_h(t, x', y) = \boldsymbol{U}\big(y + f(t, x')\big), \quad u_d(t, x', y) = -f_t(t, x') - \boldsymbol{U}\big(y + f(t, x')\big) \cdot \nabla_h f(t, x'),$$

where the function $f(t, x')$ is the solution of

$$\begin{cases} f_t + \boldsymbol{U}(f) \cdot \nabla_h f = 0, \\ f(0, x') = f_0(x'). \end{cases}$$

If

$$t^* \;:=\; -\Big[\inf_{x' \in \mathbf{R}^2} \boldsymbol{U}'\big(f_0(x')\big) \cdot \nabla_h f_0(x')\Big]^{-1} > 0,$$

then $\nabla_h \boldsymbol{u}_h$ and u_d blow up as $t \uparrow t^*$.

The proof of the above proposition is similar to the one of Proposition 3.1 in [12], so we omit it here.

2.2. **Convergence to the inviscid Prandtl equations.** In this subsection, we investigate the asymptotic behavior of solution to (1.1), as θ tends to a positive constant. We take the simple case (2.21) for brevity, and the general case can be studied by similar arguments. More precisely, we will consider the asymptotic behavior of solution under the condition

$$\theta^0(t,x') \;=\; 1+\epsilon\tilde{\theta}^0(t,x'),\quad \theta_0(x',y) \;=\; 1+\epsilon\tilde{\theta}_0(x',y) \tag{2.22}$$

with $\epsilon << 1$. Formally, (2.21) tends to the following inviscid Prandtl system for small ϵ,

$$\begin{cases} \partial_t \mathbf{u}_h + (\mathbf{u}_h\cdot\nabla_h + u_d\partial_y)\mathbf{u}_h = 0, \\ \nabla_h\cdot\mathbf{u}_h + \partial_y u_d = 0, \\ u_d|_{y=0} = 0, \quad \mathbf{u}_h|_{t=0} = \mathbf{u}_{h0}(x',y). \end{cases} \tag{2.23}$$

Through analogous arguments in Proposition 2.1, it's not difficult to show the local exsitence of solution to (2.23), where we don't need monotonicity condition on the initial data, either.

Proposition 2.3. *Let $u_0(x',y)$ be smooth initial data of problem* (2.23) *such that*

$$t_1^* := \inf\{t : \inf_{(x',y)\in\mathbb{R}^d_+} det(I_{d-1} + s\nabla_h \boldsymbol{u}_{h0}(x',y)) > 0,\ \forall s\in[0,t]\} > 0. \tag{2.24}$$

Then, (2.23) *has a unique classical solution in* $[0,t^*)$ *given by*

$$\begin{aligned} \boldsymbol{u}_h(t,x',y) =& \boldsymbol{u}_{h0}\Big(\xi_1(t,x',\eta_1(t,x',y)),\eta_1(t,x',y)\Big), \\ u_d(t,x',y) =& \int_0^{\eta_1(t,x',y)} \partial_t(\frac{1}{a_1})(t,x',z)dz + \int_0^{\eta_1(t,x',y)} \Big[b_1(t,x',\eta_1(t,x',y))\cdot\nabla_h(\frac{1}{a_1})(t,x',z)\Big]dz. \end{aligned} \tag{2.25}$$

Here, the vector function $\xi_1(t,x',z)\in\mathbb{R}^{d-1}$ is the solution of

$$x' \;=\; \xi_1 + t\boldsymbol{u}_{h0}(\xi_1,z); \tag{2.26}$$

the scale function $a_1(t,x',z)$ and vector function $b_1(t,x',z)$ are given by the initial data and $\xi_1(t,x',z)$:

$$\begin{cases} a_1(t,x',z) \;:=\; det(I_{d-1}+t\nabla_h\boldsymbol{u}_{h0})(\xi_1(t,x',z),z), \\ b_1(t,x',z) \;:=\; \boldsymbol{u}_{h0}(\xi_1(t,x',z),z); \end{cases} \tag{2.27}$$

and the scale function $\eta_1(t,x',y)$ is the solution of

$$y \;=\; \int_0^{\eta_1} \frac{1}{a_1(t,x',z)}dz. \tag{2.28}$$

Now, we show that the solution of (2.21) given in Proposition 2.1 converges to $(\mathbf{u}_h, u_d, 1)$, where $(\mathbf{u}_h, u_d)$ is a solution of (2.23) given in Proposition 2.3.

Proposition 2.4. *For the problem* (2.21) *with smooth initial-boundary data $(\boldsymbol{u}_{h0},\theta_0)$ and θ^0 satisfying* (2.22) *and the assumptions of Proposition 3.12, let $(\boldsymbol{u}_h,u_d,\theta)(t,x',y)$ be the solution of* (2.21). *Also, we assume that*

$$(1+y)^k\tilde{\theta}_0(x',y) \;\in\; H^2_{x'}(R^{d-1}, H^1_y(\mathbb{R}_+)) \tag{2.29}$$

for some constant $k>\frac{1}{2}$. Let $(\boldsymbol{u}_{h1},u_{d1})(t,x',y)$ be the solution of problem (2.23) *with smooth initial data $\boldsymbol{u}_{h0}$.*

Then, for sufficiently small ϵ there is a constant $C>0$ independent of ϵ, such that for $(t,x',y)\in[0,t_0)\times\mathbb{R}^d_+$ with t_0 being given in Proposition 2.1,

$$\big|(\boldsymbol{u}_h,u_d,\theta)(t,x',y) - (\boldsymbol{u}_{h1},u_{d1},1)(t,x',y)\big| \;\le\; C\epsilon. \tag{2.30}$$

Proof. Firstly, by (2.22) the representation of θ in (2.3) can be rewritten as

$$\theta(t,x',y) \;=\; 1+\epsilon\tilde\theta(t,x',\eta(t,x',y)), \tag{2.31}$$

where $\tilde\theta(t,x',z)$ satisfies

$$\begin{cases}\partial_t\tilde\theta+b\cdot\nabla_h\tilde\theta-a\partial_z\big(\frac{a}{1+\epsilon\tilde\theta}\partial_z\tilde\theta\big)=0, & in\ \{0\le t<t^*, z>0, x'\in\mathbb{R}^{d-1}\},\\ \tilde\theta|_{z=0}=\tilde\theta^0(t,x'), \quad \tilde\theta|_{t=0}=\tilde\theta_0(x',z). \end{cases} \tag{2.32}$$

Then, through classical Picard method we may have the local existence of solution to (2.32). Moreover, under the assumption (2.29), by the standard energy method it's not difficult to obtain that there is a constant $C_1>0$ independent of ϵ, such that

$$\|(1+z)^k\tilde\theta|_{L^\infty_t(H^2_{x'},H^1_z)} \;\le\; C_1, \tag{2.33}$$

which also shows the component of θ in (2.30) by the Sobolev embedding inequality.

Secondly, compare Propositions 2.1 with 2.4, we know that the auxilary function $\xi(t,x',z)$ given by (2.4) coincides with $\xi_1(t,x',z)$ in (2.26), which implies that by combining (2.5) with (2.27),

$$a(t,x',z) \;=\; a_1(t,x',z)\big[1+\epsilon\tilde\theta_0(\xi(t,x',z),z)\big], \quad b(t,x',z) \;=\; b_1(t,x',z). \tag{2.34}$$

Then, from (2.7) and (2.28) we have

$$\int_0^{\eta(t,x',z)}\frac{1+\epsilon\tilde\theta(t,x',z)}{a(t,x',z)}dz \;=\; \int_0^{\eta_1(t,x',z)}\frac{1}{a_1(t,x',z)}dz,$$

which implies that by (2.34),

$$\epsilon\int_0^{\eta(t,x',z)}\frac{\tilde\theta(t,x',z)-\tilde\theta_0(\xi(t,x',z),z)}{a(t,x',z)}dz \;=\; \int_{\eta(t,x',z)}^{\eta_1(t,x',z)}\frac{1}{a_1(t,x',z)}dz, \tag{2.35}$$

From (2.27) we know that the functions a and a_1 are bounded and have positive lower bounds, that is, there is a constant C_2 independent of ϵ such that

$$C_2^{-1} \;\le\; a(t,x',z),\ a_1(t,x',z) \;\le\; C_2, \quad (t,x',z)\in[0,t_0)\times\mathbb{R}^d_+,$$

then the right-hand side of the above equality gives that

$$\Big|\int_{\eta(t,x',z)}^{\eta_1(t,x',z)}\frac{1}{a_1(t,x',z)}dz\Big| \;\ge\; \frac{|\eta-\eta_1|(t,x',z)}{C_2}. \tag{2.36}$$

On the other hand, we have that for the left-hand side term of (2.38),

$$\begin{aligned}\Big|\int_0^{\eta(t,x',z)}\frac{\tilde\theta(t,x',z)-\tilde\theta_0(\xi(t,x',z),z)}{a(t,x',z)}dz\Big| &\le C_2(\|\tilde\theta(t,x',z)\|_{L^1_z}+\|\tilde\theta_0(\xi(t,x',z),z)\|_{L^1_z})\\ &\le C_3\Big(\|(1+z)^k\tilde\theta(t,x',z)\|_{L^2_z}+\|\|(1+z)^k\tilde\theta_0(\xi(t,x',z),z)\|_{L^2_z}\Big).\end{aligned}$$

Note that for $t\in[0,t_0)$ and $x'\in\mathbb{R}^{d-1}$,

$$|\tilde\theta(t,x',z)|\le\|\tilde\theta(t,x',z)\|_{H^2_{x'}},$$

and

$$|\tilde\theta_0(\xi(t,x',z),z)|\le\|\tilde\theta_0(\xi(t,x',z),z)\|_{H^2_{x'}}\le C_4\|\tilde\theta_0(x',z)\|_{H^2_{x'}},$$

where we use that $\xi(t,x',z)$ has bounded derivatives up to order two, provided that the smooth initial data $\mathbf{u}_{h0}$. From the above three expressions we obtain that

$$\Big|\int_0^{\eta(t,x',z)}\frac{\tilde\theta(t,x',z)-\tilde\theta_0(\xi(t,x',z),z)}{a(t,x',z)}dz\Big|\le C_5\|(1+z)^k\tilde\theta\|_{L^\infty_t(H^2_{x'},L^2_z)}. \tag{2.37}$$

for some constant $C_5>0$ independent of ϵ. Plugging (2.36) and (2.37) into (2.35) implies that

$$|\eta-\eta_1|(t,x',z) \;\le\; C_2C_5\|(1+z)^k\tilde\theta\|_{L^\infty_t(H^2_{x'},L^2_z)}\epsilon, \quad \forall(t,x',z)\in[0,t_0)\times\mathbb{R}^d_+. \tag{2.38}$$

Now we will prove the components of $\mathbf{u}_h$ and u_d of (2.30). Since $\xi(t,x',z) = \xi_1(t,x',z)$, it follows that from the formulas of u and u_1 given by (2.3) and (2.25) respectively,

$$|\mathbf{u}_h(t,x',y) - \mathbf{u}_{h1}(t,x',y)| \le \|\xi_z(t,x',z)\cdot\nabla_h \mathbf{u}_{h0}(\xi(t,x',z),z) + \partial_y \mathbf{u}_{h0}(\xi(t,x',z),z)\|_{L^\infty}\cdot|\eta-\eta_1|(t,x',y).$$

Combining (2.38) with the above inequality yields that there is a constant $C_6 > 0$ independent of ϵ, such that

$$|\mathbf{u}_h(t,x',y) - \mathbf{u}_{h1}(t,x',y)| \le C_6\,\epsilon, \tag{2.39}$$

provided the smooth initial data $\mathbf{u}_{h0}$. Similarly, we can show the component of u_d in (2.30), and complete the proof of this proposition. □

2.3. **The linearization around a shear flow.** In this subsection, we study the well-posedness and stability of the linearization of problem (2.21) around a shear flow. It is easy to know that under the proper initial-boundary values, (2.21) has a shear flow solution:

$$(\mathbf{u}_h, u_d, \theta)(t,x',y) = \Big(\mathbf{U}_h(y), 0, 1\Big). \tag{2.40}$$

Then, the linearization of (2.21) at the shear flow (2.40) is

$$\begin{cases} \partial_t \mathbf{u}_h + \mathbf{U}_h(y)\cdot\nabla_h \mathbf{u}_h + \mathbf{U}_h'(y) u_d = 0,\\ \partial_t\theta + \mathbf{U}_h(y)\cdot\nabla_h\theta = \partial_y^2\theta,\\ \nabla_h\cdot\mathbf{u}_h + \partial_y u_d = \partial_y^2\theta,\\ (u_d,\theta)|_{y=0} = 0, \quad (\mathbf{u}_h,\theta)|_{t=0} = (\mathbf{u}_{h0},\theta_0)(x',y). \end{cases} \tag{2.41}$$

We may solve the problem (2.41) by two steps. Firstly, we determine $\theta(t,x',y)$ through the following linear initial-boundary value problem:

$$\begin{cases} \partial_t\theta + \mathbf{U}_h(y)\cdot\nabla_h\theta = \partial_y^2\theta,\\ \theta|_{y=0} = 0, \quad \theta|_{t=0} = \theta_0(x',y). \end{cases} \tag{2.42}$$

It's easy to know that the problem (2.42) with smooth initial data has a global classical solution and the solution is unique. Secondly, with the known function $\theta(t,x',y)$ given by (2.42), we solve $(\mathbf{u}_h,u_d)(t,x',y)$ by the problem:

$$\begin{cases} \partial_t \mathbf{u}_h + \mathbf{U}_h(y)\cdot\nabla_h \mathbf{u}_h + \mathbf{U}_h'(y) u_d = 0,\\ \nabla_h\cdot\mathbf{u}_h + \partial_y u_d = \partial_y^2\theta,\\ u_d|_{y=0} = 0, \quad \mathbf{u}_h|_{t=0} = \mathbf{u}_{h0}(x',y). \end{cases} \tag{2.43}$$

Therefore, we have the following result for the problem (2.41).

Proposition 2.5. *Let $\boldsymbol{U}_h(y)$, $\boldsymbol{u}_{h0}(x',y)$ and $\theta_0(x',y)$ be smooth functions, then there exists a classical solution $(\boldsymbol{u}_h,u_d,\theta)(t,x',y)$ to the problem (2.41), where $\theta(t,x',y)$ is solved by the problem (2.42), and $(\boldsymbol{u}_h,u_d)(t,x',y)$ is given by*

(2.44)
$$\begin{aligned} \boldsymbol{u}_h(t,x',y) =& \boldsymbol{u}_{h0}(x'-t\boldsymbol{U}_h(y),y) + t\boldsymbol{U}_h'(y)\int_0^y (\nabla_h\cdot\boldsymbol{u}_{h0})(x'-t\boldsymbol{U}_h(z),z)dz\\ &+ \boldsymbol{U}_h'(y)\int_0^y \theta_0(x'-t\boldsymbol{U}_h(z),z)dz - \boldsymbol{U}_h'(y)\int_0^y\theta(t,x',z)dz,\\ u_d(t,x',y) =& -\int_0^y\Big\{(\nabla_h\cdot\boldsymbol{u}_{h0})(x'-t\boldsymbol{U}_h(z),z) + t\big[\boldsymbol{U}_h(y)-\boldsymbol{U}_h(z)\big]\cdot\nabla_h(\nabla_h\cdot\boldsymbol{u}_{h0})(x'-t\boldsymbol{U}_h(z),z)\Big\}dz\\ &-\int_0^y\Big\{\big[\boldsymbol{U}_h(y)-\boldsymbol{U}_h(z)\big]\cdot\nabla_h\theta_0(x'-t\boldsymbol{U}_h(z),z)\Big\}dz + \theta_y(t,x',y) - \theta_y(t,x',0)\\ &+\int_0^y\Big\{\big[\boldsymbol{U}_h(y)-\boldsymbol{U}_h(z)\big]\cdot\nabla_h\theta(t,x',z)\Big\}dz. \end{aligned}$$

Proof. According to the above arguments, we only need to show the expressions (2.44) of $(\mathbf{u}_h, u_d)$. From the problem (2.41), we know that $\mathbf{u}_h$ satisfy a transport equation, and from the equations of (2.41), it follows that $\partial_y u_d$ satisfies the equation

$$(\partial_t + \mathbf{U}_h(y)\cdot\nabla_h)(\partial_y u_d - \partial_y^2\theta) - (\mathbf{U}_h'(y)\cdot\nabla_h)u_d = 0, \tag{2.45}$$

and the initial data

$$\partial_y u_d(0,x',y) = \partial_y^2\theta_0(x',y) - (\nabla_h\cdot\mathbf{u}_{h0})(x',y). \tag{2.46}$$

Thereby, we introduce the following coordinate transformation:

$$\tau = t, \quad \xi = x' - t\mathbf{U}_h(y), \quad \eta = y \tag{2.47}$$

with $\xi = (\xi_1,\cdots,\xi_{d-1}) \in \mathbb{R}^{d-1}$, and obtain the corresponding partial derivatives as follows:

$$\partial_\tau = \partial_t + \mathbf{U}_h(y)\cdot\nabla_h, \quad \nabla_\xi = \nabla_h, \quad \partial_\eta = \partial_y + t\mathbf{U}_h'(y)\cdot\nabla_h. \tag{2.48}$$

Then the first two equations of (2.41) and (2.45) are reduced as

$$\begin{cases} \partial_\tau\mathbf{u}_h + \mathbf{U}_h'(\eta)u_d = 0, \quad \partial_\tau\theta = \partial_y^2\theta, \\ \partial_\tau(\partial_y u_d - \partial_y^2\theta) - \mathbf{U}_h'(\eta)\cdot\nabla_\xi u_d = 0, \end{cases} \tag{2.49}$$

where $\nabla_\xi = (\partial_{\xi_1},\cdots,\partial_{\xi_{d-1}})^T$. Moreover, we have the initial data:

$$(\mathbf{u}_h,\theta,\partial_y u_d)(0,\xi,\eta) = (\mathbf{u}_{h0},\theta_0,\partial_y^2\theta_0 - \nabla_h\cdot\mathbf{u}_{h0})(\xi,\eta). \tag{2.50}$$

Set a Lagrangian streamfunction Ψ satisfying that

$$\Psi_\eta = \partial_y u_d, \quad \Psi|_{\eta=0} = 0, \tag{2.51}$$

then, from (2.50) we have the following initial data:

$$\Psi_\eta|_{\tau=0} = (\partial_y^2\theta_0 - \nabla_h\cdot\mathbf{u}_{h0})(\xi,\eta). \tag{2.52}$$

Combining (2.51) with the third equation of (2.49), and using the second equation of (2.49) we have

$$\partial_\tau\Psi_\eta - \mathbf{U}_h'(\eta)\cdot\nabla_\xi u_d = \partial_\tau^2\theta, \tag{2.53}$$

which implies that by virtue of (2.47)

$$\partial_\tau(\tau\Psi_\eta) = \partial_\eta u_d + \tau\partial_\tau^2\theta, \quad \text{or} \quad \partial_\eta u_d = \partial_\eta(\tau\Psi)_\tau - \tau\partial_\tau^2\theta. \tag{2.54}$$

Integrating (2.54) in η and using the boundary values of u_d and Ψ, it yields that

$$\begin{aligned} u_d &= (\tau\Psi)_\tau - \tau\partial_\tau^2\Big(\int_0^\eta \theta(\tau,\xi+\tau\mathbf{U}_h(\zeta),\zeta)d\zeta\Big) \\ &= \partial_\tau\Big(\tau\Psi - \tau\partial_\tau\int_0^\eta \theta(\tau,\xi+\tau\mathbf{U}_h(\zeta),\zeta)d\zeta + \int_0^\eta \theta(\tau,\xi+\tau\mathbf{U}_h(\zeta),\zeta)d\zeta\Big). \end{aligned} \tag{2.55}$$

Then, substituting the expression (2.55) of u_d into the equation of $\mathbf{u}_h$ in (2.49) and combining with the initial data (2.50), we have

$$\mathbf{u}_h = \mathbf{u}_{h0} - \mathbf{U}_h'(\eta)\Big[\tau\Psi - \tau\partial_\tau\int_0^\eta \theta(\tau,\xi+\tau\mathbf{U}_h(\zeta),\zeta)d\zeta + \int_0^\eta \theta(\tau,\xi+\tau\mathbf{U}_h(\zeta),\zeta)d\zeta - \int_0^\eta \theta_0(\xi,\zeta)d\zeta\Big). \tag{2.56}$$

Also, plugging the expression (2.55) of u_d into (2.53), we get

$$\begin{aligned} &\partial_\tau\Big(\Psi_\eta - \tau\mathbf{U}_h'(\eta)\cdot\nabla_\xi\Psi\Big) \\ &= \partial_\tau^2\theta - \partial_\tau\Big(\tau\mathbf{U}_h'(\eta)\cdot\nabla_\xi\big(\partial_\tau\int_0^\eta \theta(\tau,\xi+\tau\mathbf{U}_h(\zeta),\zeta)d\zeta\big)\Big) + \mathbf{U}_h'(\eta)\cdot\nabla_\xi\Big(\partial_\tau\int_0^\eta \theta(\tau,\xi+\tau\mathbf{U}_h(\zeta),\zeta)d\zeta\Big) \\ &= \partial_\tau\Big[\partial_\eta\Big(\partial_\tau\int_0^\eta \theta(\tau,\xi+\tau\mathbf{U}_h(\zeta),\zeta)d\zeta\Big) - \tau\mathbf{U}_h'(\eta)\cdot\nabla_\xi\Big(\partial_\tau\int_0^\eta \theta(\tau,\xi+\tau\mathbf{U}_h(\zeta),\zeta)d\zeta\Big)\Big] \\ &\quad + \partial_\tau\Big[\mathbf{U}_h'(\eta)\cdot\nabla_\xi\Big(\int_0^\eta \theta(\tau,\xi+\tau\mathbf{U}_h(\zeta),\zeta)d\zeta\Big)\Big], \end{aligned}$$

which implies that by virtue of(2.48),

$$\partial_\tau\Big(\Psi-\partial_\tau\int_0^\eta\theta(\tau,\xi+\tau\mathbf{U}_h(\zeta),\zeta)d\zeta\Big)_y=\partial_\tau\Big[\mathbf{U}_h'(\eta)\cdot\nabla_\xi\Big(\int_0^\eta\theta(\tau,\xi+\tau\mathbf{U}_h(\zeta),\zeta)d\zeta\Big)\Big]. \tag{2.57}$$

Integrating this equation with respect to τ and using the initial data (2.52), we have

$$\begin{aligned}&\Big(\Psi-\partial_\tau\int_0^\eta\theta(\tau,\xi+\tau\mathbf{U}_h(\zeta),\zeta)d\zeta\Big)_y\\&=-(\nabla_h\cdot\mathbf{u}_{h0})(\xi,\eta)-\mathbf{U}_h'(\eta)\cdot\nabla_\xi\Big(\int_0^\eta\theta_0(\xi,\zeta)d\zeta\Big)+\mathbf{U}_h'(\eta)\cdot\nabla_\xi\Big(\int_0^\eta\theta(\tau,\xi+\tau\mathbf{U}_h(\zeta),\zeta)d\zeta\Big)\\&=-(\nabla_h\cdot\mathbf{u}_{h0})(x'-t\mathbf{U}_h(y),y)-\mathbf{U}_h'(y)\cdot\nabla_h\Big(\int_0^y\theta_0(x'-t\mathbf{U}_h(y),z)dz\Big)\\&\quad+\mathbf{U}_h'(y)\cdot\nabla_h\Big(\int_0^y\theta(t,x'-t\mathbf{U}_h(y)+t\mathbf{U}_h(z),z)dz\Big).\end{aligned}$$

Then, integrating the above quality in y we obtain that by using the boundary condition $\Psi|_{y=0}=0$,

$$\begin{aligned}&\Psi-\partial_\tau\int_0^\eta\theta(\tau,\xi+\tau\mathbf{U}_h(\zeta),\zeta)d\zeta\\&=-\int_0^y(\nabla_h\cdot\mathbf{u}_{h0})(x'-t\mathbf{U}_h(z),z)dz-\int_0^y\int_0^{\tilde y}\Big(\mathbf{U}_h'(\tilde y)\cdot\nabla_h\theta_0(x'-t\mathbf{U}_h(\tilde y),z)\Big)dzd\tilde y\\&\quad+\int_0^y\int_0^{\tilde y}\Big(\mathbf{U}_h'(\tilde y)\cdot\nabla_h\theta(t,x'-t\mathbf{U}_h(\tilde y)+t\mathbf{U}_h(z),z)\Big)dzd\tilde y,\end{aligned}$$

which implies that

$$\begin{aligned}&\tau\Big(\Psi-\partial_\tau\int_0^\eta\theta(\tau,\xi+\tau\mathbf{U}_h(\zeta),\zeta)d\zeta\Big)+\int_0^\eta\theta(\tau,\xi+\tau\mathbf{U}_h(\zeta),\zeta)d\zeta\\&=-t\int_0^y(\nabla_h\cdot\mathbf{u}_{h0})(x'-t\mathbf{U}_h(z),z)dz+\int_0^y\theta_0(x'-t\mathbf{U}_h(y),z)dz-\int_0^y\theta_0(x'-t\mathbf{U}_h(z),z)dz+\int_0^y\theta(t,x',z)dz.\end{aligned} \tag{2.58}$$

Therefore, we substitute (2.58) into (2.55) and obtain the expression of u_d :

$$\begin{aligned}u_d(t,x',y)&=-\int_0^y\Big[\nabla_h\cdot\mathbf{u}_{h0}+(\mathbf{U}_h(y)-\mathbf{U}_h(z))\cdot\nabla_h(t\nabla_h\cdot\mathbf{u}_{h0}+\theta_0)\Big](x'-t\mathbf{U}_h(z),z)dz\\&\quad+\int_0^y(\theta_t+\mathbf{U}_h(y)\cdot\nabla_h\theta)(t,x',z)dz\\&=-\int_0^y\Big[\nabla_h\cdot\mathbf{u}_{h0}+(\mathbf{U}_h(y)-\mathbf{U}_h(z))\cdot\nabla_h(t\nabla_h\cdot\mathbf{u}_{h0}+\theta_0)\Big](x'-t\mathbf{U}_h(z),z)dz\\&\quad+\int_0^y\big[(\mathbf{U}_h(y)-\mathbf{U}_h(z))\cdot\nabla_h\theta\big](t,x',z)dz+\theta_y(t,x',y)-\theta_y(t,x',0),\end{aligned} \tag{2.59}$$

where we use the equation of θ in (2.41) Meanwhile, plugging (2.58) into (2.56) yields that

$$\mathbf{u}_h(t,x',y)=\mathbf{u}_{h0}(x'-t\mathbf{U}_h(y),y)+\mathbf{U}_h'(y)\int_0^y\big[t\nabla_h\cdot\mathbf{u}_{h0}+\theta_0\big](x'-t\mathbf{U}_h(z),z)dz-\mathbf{U}_h'(y)\int_0^y\theta(t,x',z)dz.$$

Consequently, we obtain the expressions of $(\mathbf{u}_h,u_d)$ and complete this proof. □

The representation (2.44) in Proposition 2.5 shows that there is a loss of derivatives with respect to the horizontal variables x' for the solution of problem (2.41). Denote by

$$\|\mathbf{u}_h\|(t,y)\ :=\ \Big(\int_{\mathbb{R}^{d-1}}|\mathbf{u}_h(t,x',y)|^2dx'\Big)^{\frac12}, \tag{2.60}$$

and we use the following anisotropic space:

$$L^{p,q} := \{f = f(x',y) \text{ measurable}: \ \|f\|_{L^{p,q}} := \|\|f\|_{L^p(dx')}\|_{L^q(dy)} < \infty\}, \quad 1 \le p, q \le \infty,$$

$$H^{m,k} := \{f = f(x',y) \text{ measurable}: \ \|f\|_{H^{m,k}} := \Big(\sum_{|\alpha|\le m, 0\le i\le k} \|\partial_{x'}^{\alpha}\partial_y^i f\|^2_{L^2(dx'dy)}\Big)^{\frac{1}{2}} < \infty\}$$

with

$$\partial_{x'}^{\alpha} = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_{d-1}}^{\alpha_{d-1}}, \qquad \alpha = (\alpha_1, \cdots, \alpha_{d-1}), \quad |\alpha| = \alpha_1 + \cdots + \alpha_{d-1}.$$

Then, from Proposition 2.5 we have the following result.

Proposition 2.6. *Assume that $U_h(y) \in W^{2,\infty}(\mathbb{R}_+)$ is smooth and the initial data of problem (2.41) satisfies that*

$$\|u_{h0}\|(y), \ \|\nabla_h\cdot u_{h0}\|(y), \ \|\nabla_h(\nabla_h\cdot u_{h0})\|(y), \ \|\theta_0\|(y), \ \|\nabla_h\theta_0\|(y) < \infty.$$

Let $(u_h, u_d, \theta)(t, x', y)$ be the solution of (2.41). Then, there exist positive constants $M_0 = M_0(\|U_h(y)\|_{L^\infty(\mathbb{R}_+)})$ and $C_2 = C_2(\|U_h(y)\|_{W^{2\infty}(\mathbb{R}_+)})$ independent of t, such that

$$\|\theta(t,\cdot)\|_{L^2(\mathbb{R}^d_+)} \le \|\theta_0\|_{L^2(\mathbb{R}^d_+)}, \ \|\nabla_h\theta(t,\cdot)\|_{L^2(\mathbb{R}^d_+)} \le \|\nabla_h\theta_0\|_{L^2(\mathbb{R}^d_+)}, \tag{2.61}$$

and

$$\begin{aligned} &\|\theta\|(t,y), \ \|\theta_y\|(t,y) \le M_0\big(\|\theta_0\|_{H^{1,0}} + \|\theta_0\|_{H^{0,2}}\big), \ \|\nabla_h\theta\|(t,y) \le M_0\big(\|\theta_0\|_{H^{2,0}} + \|\theta_0\|_{H^{1,2}}\big),\\ &\|\partial_y^2\theta\|(t,y) \le M_1\big(\|\theta_0\|_{H^{2,0}} + \|\theta_0\|_{H^{1,2}} + \|\theta_0\|_{H^{0,4}}\big), \end{aligned} \tag{2.62}$$

moreover,

$$\begin{aligned} \|u_h\|(t,y) \le& \|u_{h0}\|(y) + t|U_h'(y)|\cdot\int_0^y \|\nabla_h\cdot u_{h0}\|(z)dz + 2\|\theta_0\|_{L^2(\mathbb{R}^d_+)}\cdot\big|\sqrt{y}U_h'(y)\big|,\\ \|u_d\|(t,y) \le& \int_0^y \|\nabla_h\cdot u_{h0}\|(z)dz + t\int_0^y\Big[\big|U_h(y) - U_h(z)\big|\cdot\|\nabla_h(\nabla_h\cdot u_{h0})\|(t,z)\Big]dz\\ &+ 2\|\nabla_h\theta_0\|_{L^2(\mathbb{R}^d_+)}\Big(\int_0^y \big|U_h(y) - U_h(z)\big|^2dz\Big)^{\frac{1}{2}} + M_0\Big(\|\theta_0\|_{H^{1,0}} + \|\theta_0\|_{H^{0,2}}\Big). \end{aligned} \tag{2.63}$$

Proof. Firstly, from Proposition 2.5 we know that $\theta(t,\xi',y)$ satisfies the linear problem (2.42). Then, it is easy to obtain that by energy estimate,

$$\frac{d}{2dt}\|\theta(t,\cdot)\|^2_{L^2(\mathbb{R}^d_+)} + \|\partial_y\theta(t,\cdot)\|^2_{L^2(\mathbb{R}^d_+)} \ = \ 0,$$

which implies that

$$\|\theta(t,\cdot)\|_{L^2(\mathbb{R}^d_+)} \ \le \ \|\theta(0,\cdot)\|_{L^2(\mathbb{R}^d_+)} \ = \ \|\theta_0\|_{L^2(\mathbb{R}^d_+)}. \tag{2.64}$$

Denote by the operator

$$\partial_{\mathcal{T}}^{\alpha} \ := \ \partial_t^{\alpha_1}\partial_{x_1}^{\alpha_2}\cdot\partial_{x_{d-1}}^{\alpha_d}, \qquad \alpha = (\alpha_1, \cdots, \alpha_d), \quad |\alpha| = \alpha_1 + \cdots + \alpha_d.$$

Applying the operator $\partial_{\mathcal{T}}^{\alpha}$, $|\alpha| = 1$ to the equation of (2.42), and similarly we have that,

$$\|\partial_{\mathcal{T}}^{\alpha}\theta(t,\cdot)\|_{L^2(\mathbb{R}^d_+)} \ \le \ \|\partial_{\mathcal{T}}^{\alpha}\theta(0,\cdot)\|_{L^2(\mathbb{R}^d_+)}. \tag{2.65}$$

Combining the equation of (2.42) with the estimates in (2.65), and using that

$$\theta_t(0,x',y) = \partial_y^2\theta_0(x',y) - \mathbf{U}_h(y)\cdot\nabla_h\theta_0(x',y), \quad \theta_{x_i}(0,x',y) = \theta_{0x_i}(x',y), \ 1 \le i \le d-1,$$

it follows that

$$\begin{aligned} \|\partial_y^2\theta(t,\cdot)\|_{L^2(\mathbb{R}^d_+)} \ \le& \ \|\theta_t(t,\cdot)\|_{L^2(\mathbb{R}^d_+)} + \|\mathbf{U}_h(y)\|_{L^\infty(\mathbb{R}_+)}\cdot\|\nabla_h\theta(t,\cdot)\|_{L^2(\mathbb{R}^d_+)}\\ \le& \ 2\|\mathbf{U}_h(y)\|_{L^\infty(\mathbb{R}_+)}\cdot\|\nabla_h\theta_0\|_{L^2(\mathbb{R}^d_+)} + \|\partial_y^2\theta_0\|_{L^2(\mathbb{R}^d_+)}. \end{aligned} \tag{2.66}$$

By the classical interpolation inequality we obtain that from (2.64) and (2.66),

$$\begin{aligned} \|\theta_y(t,\cdot)\|_{L^2(\mathbb{R}^d_+)} &\le C\Big(\|\theta(t,\cdot)\|_{L^2(\mathbb{R}^d_+)} + \|\partial_y^2\theta(t,\cdot)\|_{L^2(\mathbb{R}^d_+)}\Big) \\ &\le C\Big(\|\theta_0\|_{L^2(\mathbb{R}^d_+)} + \|\mathbf{U}_h(y)\|_{L^\infty(\mathbb{R}_+)}\cdot\|\nabla_h\theta_0\|_{L^2(\mathbb{R}^d_+)} + \|\partial_y^2\theta_0\|_{L^2(\mathbb{R}^d_+)}\Big), \end{aligned} \tag{2.67}$$

where C is a positive constant independent of t. Then, from the estimates (2.64), (2.66) and (2.67) it implies by the imbedding inequality that there is a positive constant $C_0 = C_0(\|\mathbf{U}_h(y)\|_{L^\infty(\mathbb{R}_+)})$ independent of t, such that

$$\begin{aligned} \|\theta(t,\cdot)\|_{L^{2,\infty}} &\le \|\theta(t,\cdot)\|_{H^{0,1}} \le C_0\big(\|\theta_0\|_{H^{1,0}} + \|\theta_0\|_{H^{0,2}}\big), \\ \|\theta_y(t,\cdot)\|_{L^{2,\infty}} &\le \|\theta_y(t,\cdot)\|_{H^{0,1}} \le C_0\big(\|\theta_0\|_{H^{1,0}} + \|\theta_0\|_{H^{0,2}}\big). \end{aligned} \tag{2.68}$$

Next, we apply ∂_T^α, $|\alpha| = 1$ to the equation in (2.42) and get

$$\partial_t\partial_T^\alpha\theta + \mathbf{U}_h(y)\cdot\nabla_h\partial_T^\alpha\theta - \partial_y^2\partial_T^\alpha\theta = 0,$$

moreover, we have the following boundary value of $\partial_T^\alpha\theta(t,x',y)$:

$$\partial_T^\alpha\theta_{x_i}(t,x',0) \;=\; 0.$$

Thus, through the above analogous arguments we can obtain that there exist positive constants $C_1 = C_1(\|\mathbf{U}_h(y)\|_{L^\infty(\mathbb{R}_+)})$ and $C_2 = C_2(\|\mathbf{U}_h(y)\|_{W^{2\infty}(\mathbb{R}_+)})$ independent of t, such that

$$\begin{aligned} \|\nabla_h\theta(t,\cdot)\|_{L^{2,\infty}} &\le C_1\big(\|\theta_0\|_{H^{2,0}} + \|\theta_0\|_{H^{1,2}}\big), \\ \|\theta_t(t,\cdot)\|_{L^{2,\infty}} &\le C_2\big(\|\theta_0\|_{H^{2,0}} + \|\theta_0\|_{H^{1,2}} + \|\theta_0\|_{H^{0,4}}\big). \end{aligned} \tag{2.69}$$

Thus, from the equation in (2.42) we obtain that there is a positive constant $C_3 = C_3(\|\mathbf{U}_h(y)\|_{W^{2\infty}(\mathbb{R}_+)})$ independent of t, such that

$$\begin{aligned} \|\partial_y^2\theta(t,\cdot)\|_{L^{2,\infty}} &\le \|\theta_t(t,\cdot)\|_{L^{2,\infty}} + \|\mathbf{U}_h(y)\|_{L^\infty(\mathbb{R}_+)}\cdot\|\nabla_h\theta\|_{L^{2,\infty}} \\ &\le C_3\big(\|\theta_0\|_{H^{2,0}} + \|\theta_0\|_{H^{1,2}} + \|\theta_0\|_{H^{0,4}}\big). \end{aligned} \tag{2.70}$$

Combining (2.68), (2.69) and (2.70), we obtain the estimates (2.62).

Finally, it is easy to establish the following estimates for $(\mathbf{u}_h, u_d)(t,x',y)$ by the expression (2.44) in Proposition 2.5,

$$\begin{aligned} \|\mathbf{u}_h\|(t,y) &\le \|\mathbf{u}_{h0}\|(y) + |\mathbf{U}_h'(y)|\int_0^y\Big[\|\theta_0\|(z) + \|\theta\|(t,z)\Big]dz + t|\mathbf{U}_h'(y)|\int_0^y\|\nabla_h\cdot\mathbf{u}_{h0}\|(z)dz \\ &\le \|\mathbf{u}_{h0}\|(y) + t|\mathbf{U}_h'(y)|\int_0^y\|\nabla_h\cdot\mathbf{u}_{h0}\|(z)dz + \big|\sqrt{y}\,\mathbf{U}_h'(y)\big|\Big(\|\theta_0\|_{L^2(\mathbb{R}^d_+)} + \|\theta(t,\cdot)\|_{L^2(\mathbb{R}^d_+)}\Big), \end{aligned} \tag{2.71}$$

and

$$\begin{aligned} \|u_d\|(t,y) &\le 2\|\theta_y(t,\cdot)\|_{L^{2,\infty}} + \int_0^y\Big[\big|\mathbf{U}_h(y)-\mathbf{U}_h(z)\big|\cdot\Big(\|\nabla_h\theta_0\|(z) + \|\nabla_h\theta\|(t,z)\Big)\Big]dz \\ &\quad + \int_0^y\|\nabla_h\cdot\mathbf{u}_{h0}\|(z)dz + t\int_0^y\Big[\big|\mathbf{U}_h(y)-\mathbf{U}_h(z)\big|\cdot\|\nabla_h(\nabla_h\cdot\mathbf{u}_{h0})\|(t,z)\Big]dz \\ &\le 2\|\theta_y(t,\cdot)\|_{L^{2,\infty}} + \Big(\int_0^y\big|\mathbf{U}_h(y)-\mathbf{U}_h(z)\big|^2dz\Big)^{\frac12}\cdot\Big(\|\nabla_h\theta_0\|_{L^2(\mathbb{R}^d_+)} + \|\nabla_h\theta(t,\cdot)\|_{L^2(\mathbb{R}^d_+)}\Big) \\ &\quad + \int_0^y\|\nabla_h\cdot\mathbf{u}_{h0}\|(z)dz + t\int_0^y\Big[\big|\mathbf{U}_h(y)-\mathbf{U}_h(z)\big|\cdot\|\nabla_h(\nabla_h\cdot\mathbf{u}_{h0})\|(t,z)\Big]dz. \end{aligned} \tag{2.72}$$

Combining (2.64) with (2.71), we obtain the estimate for $\mathbf{u}_h$ in (2.63). Substituting (2.62) and (2.65) into (2.72), the estimate for u_d in (2.63) follows immediately. □

Now, we show that under some certain conditions of $\mathbf{U}_h(y)$, the shear flow solution $\big(\mathbf{U}_h(y), 0, 1\big)$ to problem (2.21) is linearly unstable, that is, solutions to the linearized problem (2.41) of (2.21) grow algebraically in time. More precisely, we have the following result:

Proposition 2.7. *Suppose that for the problem* (2.41), $\boldsymbol{U}_h(y)$ *and the initial data are smooth, and* $\boldsymbol{u}_h(x',y)$ *decays rapidly in* x'. *Let* $(\boldsymbol{u}_h, u_d, \theta)(t,x,y)$ *be the solution of* (2.41).

(1) For $d=2$, *if* $U(y)$ *has no critical points, then* $\|\boldsymbol{u}_h\|(t,y)$ *is bounded uniformly in* t:

$$\|\boldsymbol{u}_h\|(t,y) \le \Big|\frac{\boldsymbol{U}_h(y)}{\boldsymbol{U}_h'(0)}\Big| \cdot \|\boldsymbol{u}_{h0}\|(0) + \int_0^y \Big\{\Big|\frac{\boldsymbol{U}_h(y)}{\boldsymbol{U}_h'(z)}\Big| \cdot \|\partial_y \boldsymbol{u}_{h0}\|(z) + \Big|\frac{\boldsymbol{U}_h''(z)}{(\boldsymbol{U}_h'(z))^2}\Big| \cdot \|\boldsymbol{u}_{h0}\|(z)\Big\} dz + 2\|\theta_0\|_{L^2(\mathbf{R}^d_+)}\sqrt{y}\,|\boldsymbol{U}_h'(y)|, \tag{2.73}$$

and as $t \to +\infty$,

$$\Big\|\boldsymbol{u}_h + \boldsymbol{U}_h'(y)\int_0^y \theta(t,x',z)dz\Big\|(t,y) \to \|\boldsymbol{u}_{h0}\|(0) \cdot \Big|\frac{\boldsymbol{U}_h(y)}{\boldsymbol{U}_h'(0)}\Big|. \tag{2.74}$$

If $U(y)$ *has a single, non-degenerate critical point at* $y=y_0$, *then when* $y>y_0$, *we have that when* $t\to+\infty$,

$$\Big\|\boldsymbol{u}_h + \boldsymbol{U}_h'(y)\int_0^y \theta(t,x',z)dz\Big\|(t,y) \sim \sqrt{2\pi t}\,\|\boldsymbol{u}_{h0}\|_{\frac12}(y_0)\frac{|\boldsymbol{U}_h'(y)|}{\sqrt{|\boldsymbol{U}_h''(y_0)|}}, \tag{2.75}$$

where

$$\|\boldsymbol{u}_{h0}\|_{\frac12}(y) := \Big(\int_{\mathbf{R}^2} |\xi| \cdot |\hat{\boldsymbol{u}}_{h0}|^2(\xi,y)d\xi\Big)^{\frac12}.$$

Also, for sufficiently large t *we have*

$$\|\boldsymbol{u}_h\|(t,y) \ge \sqrt{\frac{\pi t}{2}}\,\|\boldsymbol{u}_{h0}\|_{\frac12}(y_0)\frac{|\boldsymbol{U}_h'(y)|}{\sqrt{|U_1''(y_0)|}}. \tag{2.76}$$

Furthermore, we have similar results as above for $\partial_y \boldsymbol{u}_h$, u_d *and* $\partial_y u_d$.

(2) For $d=3$, *suppose that* $\boldsymbol{U}_h(y)$ *satisfies that* $U_i(y)$ $(i=1$ *or* $2)$ *has no critical points (we assume that* $i=1$*), and* $U_2(y)=kU_1(y)$ *for some constant* k. *Then* $\|u_d\|(t,y)$ *is bounded uniformly in* t:

$$\begin{aligned}\|u_d\|(t,y) \le & \Big|\frac{U_1(y)-U_1(0)}{U_1'(0)}\Big| \cdot \|\nabla_h \cdot \boldsymbol{u}_{h0}\|(0) + \int_0^y \Big\{\Big|\frac{U_1(y)-U_1(z)}{U_1'(z)}\Big| \cdot \|\nabla_h \cdot \partial_y \boldsymbol{u}_{h0}\|(z) \\ & + \Big|\frac{U_1''(z)(U_1(y)-U_1(z))}{(U_1'(z))^2}\Big| \cdot \|\nabla_h \cdot \boldsymbol{u}_{h0}\|(z)\Big\}dz \\ & + 2\|\nabla_h\theta_0\|_{L^2(\mathbf{R}^d_+)}\Big(\int_0^y |\boldsymbol{U}_h(y)-\boldsymbol{U}_h(z)|^2dz\Big)^{\frac12} + M_0\Big(\|\theta_0\|_{H^{1,0}} + \|\theta_0\|_{H^{0,2}}\Big).\end{aligned} \tag{2.77}$$

Moreover, as $t\to+\infty$,

$$\Big\|u_d - \theta_y(t,x',y) + \theta_y(t,x',0) - \int_0^y \big[(\boldsymbol{U}_h(y)-\boldsymbol{U}_h(z))\cdot\nabla_h\theta(t,x',z)\big]dz\Big\|(t,y) \to \|\nabla_h\cdot\boldsymbol{u}_{h0}\|(0)\Big|\frac{U_1(y)-U_1(0)}{U_1'(0)}\Big|. \tag{2.78}$$

Assume that $U_i(y)$ $(i=1$ *or* $2)$ *(we assume that* $i=1$*) has a single, non-degenerate critical point at* $y=y_0$, *and* $U_2(y)=kU_1(y)$ *for some constant* k. *Then when* $y>y_0$, *we have that when* $t\to+\infty$:

$$\begin{aligned}&\Big\|u_d - \theta_y(t,x',y) + \theta_y(t,x',0) - \int_0^y \big[(\boldsymbol{U}_h(y)-\boldsymbol{U}_h(z))\cdot\nabla_h\theta(t,x',z)\big]dz\Big\|(t,y) \\ &\sim \sqrt{2\pi t}\,\|\nabla_h\cdot\boldsymbol{u}_{h0}\|_{\frac12,k}(y_0)\frac{|U_1(y)-U_1(y_0)|}{\sqrt{|U_1''(y_0)|}},\end{aligned} \tag{2.79}$$

where

$$\|\nabla_h\cdot\boldsymbol{u}_{h0}\|_{\frac12,k}(y) := \Big(\int_{\mathbf{R}^2} |\xi_1+k\xi_2||\xi\cdot\hat{\boldsymbol{u}}_{h0}|^2(\xi,y)d\xi\Big)^{\frac12}.$$

Also, for sufficiently large t *we have*

$$\|u_d\|(t,y) \ge \sqrt{\frac{\pi t}{2}}\,\|\nabla_h\cdot\boldsymbol{u}_{h0}\|_{\frac12,k}(y_0)\frac{|U_1(y)-U_1(y_0)|}{\sqrt{|U_1''(y_0)|}}. \tag{2.80}$$

If $U_2(y)\neq kU_1(y)$ for any $k\in\mathbb{R}$, then there is a point y_0 such that, when $y>y_0$ we have that for sufficiently large t,

$$\|u_d\|(t,y)\ \geq\ C\sqrt{t} \tag{2.81}$$

with the constant $C=C\big(y,y_0,\boldsymbol{U}_h,\boldsymbol{u}_{h0}\big)>0$ independent of t. Moreover, we have similar results as above for $\partial_y u_d$ and $\nabla_h\cdot\boldsymbol{u}_h$.

Proof. We will prove the second part of this proposition, the first part of two-dimensional case is similar to Proposition 6.1 of [12]. From Proposition 2.5, we have the expressions (2.44) of the solution to problem (2.41), then we take the Fourier transform with respect to x' for the expression of $u_d(t,x',y)$ to obtain that

$$\begin{aligned}\hat{u}_d(t,\xi,y)=&-\int_0^y\Big\{i\xi\cdot\hat{\mathbf{u}}_{h0}(\xi,y)+\big[i\xi\cdot(\mathbf{U}_h(y)-\mathbf{U}_h(z))\big]\cdot\big[it\xi\cdot\hat{\mathbf{u}}_{h0}(\xi,z)+\hat{\theta}_0(\xi,z)\big]\Big\}e^{-it\xi\cdot\mathbf{U}_h(z)}dz\\&+\int_0^y\big[i\xi\cdot(\mathbf{U}_h(y)-\mathbf{U}_h(z))\big]\hat{\theta}(t,\xi,z)dz+\hat{\theta}_y(t,\xi,y)-\hat{\theta}_y(t,\xi,0).\end{aligned} \tag{2.82}$$

When $U_2(y)=kU_1(y)$ for some constant k, then (2.82) is reduced as

$$\begin{aligned}\hat{u}_d(t,\xi,y)=&-\int_0^y\big[1+it(\xi_1+k\xi_2)(U_1(y)-U_1(z))\big]\big[i\xi\cdot\hat{\mathbf{u}}_{h0}(\xi,z)\big]e^{-it(\xi_1+k\xi_2)U_1(z)}dz\\&-\int_0^y\big[i\xi\cdot(\mathbf{U}_h(y)-\mathbf{U}_h(z))\big]\cdot\big[\hat{\theta}_0(\xi,z)e^{-it\xi\cdot\mathbf{U}_h(z)}+\hat{\theta}(t,\xi,z)\big]dz+\hat{\theta}_y(t,\xi,y)-\hat{\theta}_y(t,\xi,0).\end{aligned} \tag{2.83}$$

If $U_i(y)$ (we may assume that $i=1$) has no critical points and for $\xi_1\neq-k\xi_2$ in the above equality, we obtain that by integration by parts,

$$\begin{aligned}\hat{u}_d(t,\xi,y)=&-\frac{U_1(y)-U_1(0)}{U_1'(0)}\big[i\xi\cdot\hat{\mathbf{u}}_{h0}(\xi,0)\big]e^{-it(\xi_1+k\xi_2)U_1(0)}-\int_0^y\Big\{\frac{U_1(y)-U_1(z)}{U_1'(z)}\\&\big[i\xi\cdot\widehat{\partial_y\mathbf{u}}_{h0}(\xi,z)\big]-\frac{U_1''(z)\big(U_1(y)-U_1(z)\big)}{\big(U_1'(z)\big)^2}\big[i\xi\cdot\hat{\mathbf{u}}_{h0}(\xi,z)\big]\Big\}e^{-it(\xi_1+k\xi_2)U_1(z)}dz\\&-\int_0^y\big[i\xi\cdot(\mathbf{U}_h(y)-\mathbf{U}_h(z))\big]\cdot\big[\hat{\theta}_0(\xi,z)e^{-it\xi\cdot\mathbf{U}_h(z)}-\hat{\theta}(t,\xi,z)\big]dz\\&+\hat{\theta}_y(t,\xi,y)-\hat{\theta}_y(t,\xi,0).\end{aligned} \tag{2.84}$$

Then, from the above equality (2.84) it implies that by Parseval's identity,

$$\begin{aligned}\|u_d\|(t,y)\leq&\Big|\frac{U_1(y)-U_1(0)}{U_1'(0)}\Big|\cdot\|\nabla_h\cdot\mathbf{u}_{h0}\|(0)+\int_0^y\Big[\Big|\frac{U_1(y)-U_1(z)}{U_1'(z)}\Big|\cdot\|\nabla_h\cdot\partial_y\mathbf{u}_{h0}\|(z)\\&+\Big|\frac{U_1''(z)\big(U_1(y)-U_1(z)\big)}{\big(U_1'(z)\big)^2}\Big|\cdot\|\nabla_h\cdot\mathbf{u}_{h0}\|(z)\Big]dz\\&+\int_0^y\big|\mathbf{U}_h(y)-\mathbf{U}_h(z)\big|\cdot\big[\|\nabla_h\theta_0\|(z)+\|\nabla_h\theta\|(t,z)\big]dz+2\|\theta_y(t,\cdot)\|_{L^{2,\infty}},\end{aligned} \tag{2.85}$$

thus, by using Proposition 2.6 in the above estimate (2.85) we obtain (2.77). Moreover, from (2.84) we have

$$\begin{aligned}&\hat{u}_d(t,\xi,y)-\int_0^y\big[i\xi\cdot(\mathbf{U}_h(y)-\mathbf{U}_h(z))\big]\cdot\hat{\theta}(t,\xi,z)dz-\hat{\theta}_y(t,\xi,y)+\hat{\theta}_y(t,\xi,0)\\&\to-\frac{U_1(y)-U_1(0)}{U_1'(0)}\big[i\xi\cdot\hat{\mathbf{u}}_{h0}(\xi,0)\big]e^{-it(\xi_1+k\xi_2)U_1(0)},\qquad\text{as } t\to+\infty.\end{aligned} \tag{2.86}$$

Thereby, (2.78) follows from (2.86) and Parseval's identity.

If $U_2(y)=kU_1(y)$ for some constant k and $U_i(y)$ (we may assume that $i=1$) has a non-degenerate critical point at $y=y_0$, then for $y>y_0$ and $\xi_1\neq-k\xi_2$, by an application of the method of stationary phase to

(2.83) yields that when $t \to +\infty$,

$$\begin{aligned}&\hat{u}_d(t,\xi,y) - \int_0^y [i\xi \cdot (\mathbf{U}_h(y) - \mathbf{U}_h(z))] \cdot \hat{\theta}(t,\xi,z)dz - \hat{\theta}_y(t,\xi,y) + \hat{\theta}_y(t,\xi,0)\\ &\sim sgn(\xi_1 + k\xi_2)\sqrt{2\pi t|\xi_1 + k\xi_2|} \cdot [\xi \cdot \hat{\mathbf{u}}_{h0}(\xi, y_0)] \frac{U_1(y) - U_1(y_0)}{\sqrt{|U_1''(y_0)|}}\\ &\cdot \exp\Big\{ - it(\xi_1 + k\xi_2)U_1(y_0) - i\ sgn\big((\xi_1 + k\xi_2)U_1''(y_0)\big)\pi/4\Big\}.\end{aligned} \tag{2.87}$$

By using Parseval's identity in the above equality, we obtain (2.79). Then, from the uniform boundedness of $\|\theta_y\|(t,y)$ and $\|\nabla_h\theta\|(t,y)$ with respect to t in Proposition 2.6, we obtain (2.80) for sufficently large t.

If $U_2(y) \neq kU_1(y)$ for any $k \in \mathbb{R}$, then there is a point y_0 such that

$$U_1'(y_0)U_2''(y_0) \neq U_2'(y_0)U_1''(y_0). \tag{2.88}$$

We may assume that $U_1'(y_0) > 0$ and $U_1'(y_0)U_2''(y_0) - U_2'(y_0)U_1''(y_0) > 0$. Then, we affirm that for any $\delta > 0$, there is a interval $S_\delta \subseteq (y_0 - \delta, y_0 + \delta)$ such that

$$U_1'(y) > 0, \quad U_2'(y) \neq 0, \quad U_1'(y)U_2''(y) - U_2'(y)U_1''(y) > 0, \qquad \forall\, y \in S_\delta. \tag{2.89}$$

Indeed, from (2.88) and the smoothness of $U_1(y), U_2(y)$ we know that there is a $\delta_0 > 0$ such that for any $y \in (y_0 - \delta_0, y_0 + \delta_0)$,

$$U_1'(y) > 0, \qquad U_1'(y)U_2''(y) - U_2'(y)U_1''(y) > 0.$$

Then, by virtue of (2.88) again we have $U_2(y_1) \neq 0$ for some $y_1 \in (y_0 - \delta_0, y_0 + \delta_0)$, thus we take some small neighborhood of y_1 as the required set S_δ. Moreover, (2.89) implies that the functions $\frac{U_2'(y)}{U_1'(y)}$ is monotonically increasing in S_δ. Next, denote by

$$I_\delta^R := \Big\{\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}; \quad |\xi| \leq R, \text{ and } \exists y \in S_\delta,\ s.t.\ \xi \cdot \mathbf{U}_h'(y) = 0\Big\}, \tag{2.90}$$

and from the monotonicity of $\frac{U_2'(y)}{U_1'(y)}$ in S_δ, we know that the point $y \in S_\delta$ satisfying $\xi \cdot \mathbf{U}_h'(y) = 0$ for $\xi \in I_\delta^R$ is unique. Moreover, by virtue of the continuity of $\mathbf{U}_h'(y)$, it is easy to know that the measure of I_δ^R is positive, i.e., $m(I_\delta^R) > 0$. For $y > y_0$ and any $\xi \in I_\delta^R$ with $\delta \leq y - y_0$, there exists a unique $y_\xi \in S_\delta$ such that $\xi \cdot \mathbf{U}_h'(y_\xi) = 0$, and then, from (2.89) we have $\xi \cdot \mathbf{U}_h''(y_\xi) \neq 0$. For such (ξ, y) in (2.82) it yields that by an application of the method of stationary phase,

$$\begin{aligned}&\hat{u}_d(t,\xi,y) - \int_0^y [i\xi \cdot (\mathbf{U}_h(y) - \mathbf{U}_h(z))] \cdot \hat{\theta}(t,\xi,z)dz - \hat{\theta}_y(t,\xi,y) + \hat{\theta}_y(t,\xi,0)\\ &\sim \sqrt{\frac{2\pi t}{|\xi \cdot \mathbf{U}_h''(y_\xi)|}} \cdot [\xi \cdot (\mathbf{U}_h(y) - \mathbf{U}_h(y_\xi))]\,[\xi \cdot \hat{\mathbf{u}}_{h0}(\xi, y_\xi)]\end{aligned} \tag{2.91}$$

Note that when δ is small enough, we have that for any $\xi \in I_\delta^R$,

$$\Big|\frac{\xi \cdot (\mathbf{U}_h(y) - \mathbf{U}_h(y_\xi))}{\sqrt{|\xi \cdot \mathbf{U}_h''(y_\xi)|}}[\xi \cdot \hat{\mathbf{u}}_{h0}(\xi, y_\xi)]\Big| \geq \Big|\frac{\xi \cdot (\mathbf{U}_h(y) - \mathbf{U}_h(y_0))}{2\sqrt{|\xi \cdot \mathbf{U}_h''(y_0)|}}[\xi \cdot \hat{\mathbf{u}}_{h0}(\xi, y_0)]\Big|.$$

Thus, for sufficiently large t we obtain that by using Parseval's identity in (2.91),

$$\begin{aligned}&\Big\|u_d - \theta_y(t,x',y) + \theta_y(t,x',0) - \int_0^y \big[(\mathbf{U}_h(y) - \mathbf{U}_h(z)) \cdot \nabla_h\theta(t,x',z)\big]dz\Big\|(t,y)\\ &\geq \frac{\sqrt{2\pi t}}{4}\Big\|\frac{\xi \cdot (\mathbf{U}_h(y) - \mathbf{U}_h(y_0))}{\sqrt{|\xi \cdot \mathbf{U}_h''(y_0)|}}[\xi \cdot \hat{\mathbf{u}}_{h0}(\xi, y_0)]\Big\|_{L^2_\xi(I_\delta^R)},\end{aligned}$$

and then, combining with the uniform boundedness of $\|\theta_y\|(t,y)$ and $\|\nabla_h\theta\|(t,y)$ with respect to t in Proposition 2.6, it implies that

$$\|u_d\|(t,y) \geq \frac{\sqrt{2\pi t}}{8}\Big\|\frac{\xi\cdot(\mathbf{U}_h(y)-\mathbf{U}_h(y_0))}{\sqrt{|\xi\cdot\mathbf{U}_h''(y_0)|}}[\xi\cdot\hat{\mathbf{u}}_{h0}(\xi,y_0)]\Big\|_{L^2_\xi(I^R_\delta)}.$$

Consequently, we get the estimate (2.81). Through analogous arguments as above, we can obtain similar results for u_d and $\nabla_h\cdot\mathbf{u}_h$ as the ones of u_d.

□

3. Appendix

Now, we will give a formal derivative of boundary layer problem (1.1) of compressible flow. Consider the following problem in the domain $\mathbb{R}_+\times\mathbb{R}^d_+$ with $d=2,3$,

$$\begin{cases}\partial_t\rho+\nabla\cdot(\rho\mathbf{u})=0,\\ \rho\{\partial_t\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}\}+\nabla p(\rho,\theta)=0,\\ c_V\rho\{\partial_t\theta+(\mathbf{u}\cdot\nabla)\theta\}+p(\rho,\theta)\nabla\cdot\mathbf{u}=\epsilon\Delta\theta,\end{cases}\tag{3.1}$$

where $t>0$, the spatial variables $x=(x',x_d)\in\mathbb{R}^d_+$ with $x'=(x_1,\cdots,x_{d-1})\in\mathbb{R}^{d-1}$ and the scale variable $x_d>0$, ρ is the density, $\mathbf{u}=(u_1,\cdots,u_d)^T$ is the velocity, θ is the absolute temperature, $p(\rho,\theta)$ is the pressure, the constant $c_V>0$ is the specific heat capacity, $\epsilon=\epsilon(\rho,\theta)$ is the coefficient of heat conduction. For the equations (3.1), we endow they with the following boundary values:

$$u_d|_{x_d=0}=0,\quad \big[\alpha\partial_{x_d}\theta+\beta\theta\big]\big|_{x_d=0}=\gamma,\tag{3.2}$$

where $\alpha=\alpha(t,x'),\beta=\beta(t,x')$ and $\gamma=\gamma(t,x')$ are given functions. In the paper, we consider the ideal gas model for the problem (3.1)-(3.2),

$$p(\rho,\theta)=R\rho\theta\tag{3.3}$$

with a positive constant R. We are concerned with the asymptotic behavior of solutions $(\rho,\mathbf{u},\theta)(t,x)$ to the problem (3.1)-(3.2) when the heat conduction coefficient tends zero, i.e., $\epsilon\to0$.

Formally, we can obtain that when $\epsilon\to0$, solutions $(\rho,\mathbf{u},\theta)(t,x)$ to problem (3.1)-(3.2) tend to $(\rho^e,\mathbf{u}^e,\theta^e)(t,x)$, which satisfy the following compressible non-isentropic Euler equations in $\mathbb{R}_+\times\mathbb{R}^d_+$:

$$\begin{cases}\partial_t\rho^e+\nabla\cdot(\rho^e\mathbf{u}^e)=0,\\ \rho^e\{\partial_t\mathbf{u}^e+(\mathbf{u}^e\cdot\nabla)\mathbf{u}^e\}+R\nabla(\rho^e\theta^e)=0,\\ c_V\rho^e\{\partial_t\theta^e+(\mathbf{u}^e\cdot\nabla)\theta^e\}+R\rho^e\theta^e(\nabla\cdot\mathbf{u}^e)=0\end{cases}\tag{3.4}$$

with the boundary condition

$$u^e_d|_{x_d=0}=0.\tag{3.5}$$

So, the inconsistent of boundary conditions between (3.2) and (3.5) leads to the appearance of boundary layer. Since the diffusion terms are important in the boundary layer and should be balanced by the convective terms, and note that the vertical component of velocity field vanishes at the boundary in the problem (3.1)-(3.2), we may just consider the characteristic boundary layers, that is, the sizes of boundary layers are $\sqrt{\epsilon}$. Therefore, we express solutions to (3.1)-(3.2) via $(\rho^\epsilon,\mathbf{u}^\epsilon,\theta^\epsilon)$ as

$$\begin{aligned}(\rho,\mathbf{u},\theta)(t,x)&=\Big(\rho(t,x',\frac{x_d}{\sqrt{\epsilon}}),\mathbf{u}_h(t,x',\frac{x_d}{\sqrt{\epsilon}}),\sqrt{\epsilon}\{\frac{u_d(t,x',\frac{x_d}{\sqrt{\epsilon}})}{\sqrt{\epsilon}}\},\theta(t,x',\frac{x_d}{\sqrt{\epsilon}})\Big)\\ &=\Big(\rho^\epsilon,\mathbf{u}^\epsilon_h,u^\epsilon_d,\theta^\epsilon\Big)(t,x',y),\end{aligned}$$

where we introduce the scale variable $y=\frac{x_d}{\sqrt{\epsilon}}$, the tangential component $\mathbf{u}_h=(u_1,\cdots,u_{d-1})$ of the velocity field $\mathbf{u}$ and note that the scale normal velocity u^ϵ_d is $\frac{1}{\sqrt{\epsilon}}$ of the original velocity u_d. In these new variables above, the problem (3.1)-(3.2) reads that in the domain $\{(t,x',y) : t>0, x'\in\mathbb{R}^{d-1}, y>0\}$:

$$
\begin{cases}
\partial_t\rho^\epsilon + \nabla_h\cdot(\rho^\epsilon \mathbf{u}_h^\epsilon) + \partial_y(\rho^\epsilon u_d^\epsilon) = 0,\\
\rho^\epsilon\{\partial_t \mathbf{u}_h^\epsilon + (\mathbf{u}_h^\epsilon\cdot\nabla_h + u_d^\epsilon\partial_y)\mathbf{u}_h^\epsilon\} + R\nabla_h(\rho^\epsilon\theta^\epsilon) = 0,\\
\rho^\epsilon\{\partial_t u_d^\epsilon + (\mathbf{u}_h^\epsilon\cdot\nabla_h + u_d^\epsilon\partial_y)u_d^\epsilon\} + \frac{R\partial_y(\rho^\epsilon\theta^\epsilon)}{\epsilon} = 0,\\
c_V\rho^\epsilon\{\partial_t\theta^\epsilon + (\mathbf{u}_h^\epsilon\cdot\nabla_h + u_d^\epsilon\partial_y)\theta^\epsilon\} + R\rho^\epsilon\theta^\epsilon(\nabla_h\cdot\mathbf{u}_h^\epsilon + \partial_y u_d^\epsilon) = \epsilon\Delta_h\theta^\epsilon + \partial_y^2\theta^\epsilon,\\
u_d^\epsilon|_{y=0} = 0, \ \left[\frac{a}{\sqrt{\epsilon}}\partial_y\theta^\epsilon + b\theta^\epsilon\right]|_{y=0} = c,
\end{cases}
\tag{3.6}
$$

where the derivatives $\nabla_h = (\partial_{x_1},\cdots,\partial_{x_{d-1}})^T$ and $\Delta_h = \partial_{x_1}^2+\cdots+\partial_{x_{d-1}}^2$.

Similar as the hypothesis of Prandtl boundary layers for the incompressible flow, we assume that solutions of (3.6) can be approximated as follows:

$$
(\rho^\epsilon, \mathbf{u}_h^\epsilon, u_d^\epsilon, \theta^\epsilon)(t,x',y) \approx (\rho^e, \mathbf{u}_h^e, \frac{u_d^e}{\sqrt{\epsilon}}, \theta^e)(t,x',\sqrt{\epsilon}y) + (\rho^b, \mathbf{u}_h^b, u_d^b, \theta^b)(t,x',y),
\tag{3.7}
$$

where $(\rho^e,\mathbf{u}^e,\theta^e)(t,x)$ denotes the Euler flow given by (3.4)-(3.5) with $\mathbf{u}^e = (\mathbf{u}_h^e, u_d^e)^T$, and the boundary layer profile $(\rho^b,\mathbf{u}_h^b,u_d^b,\theta^b)(t,x',y)$ decrease rapidly as $y\to+\infty$. We plug the ansatz (3.7) into the problem (3.6) and take the leading terms with respect to ϵ. By virtue of (3.5) and then the asymptotic expansion for the Euler flow

$$
(\rho^e, \mathbf{u}_h^e, \frac{u_d^e}{\sqrt{\epsilon}}, \theta^e)(t,x',\sqrt{\epsilon}y) \ = \ (\rho^e, \mathbf{u}_h^e, y\partial_{x_d}u_d^e, \theta^e)(t,x',0) + O(\sqrt{\epsilon}),
$$

we obtain that the new boundary layer profile

$$
(\rho, \mathbf{u}_h, u_d, \theta)(t,x',y) \ := \ (\rho^e, \mathbf{u}_h^e, y\partial_{x_d}u_d^e, \theta^e)(t,x',0) + (\rho^b, \mathbf{u}_h^b, u_d^b, \theta^b)(t,x',y)
$$

satisfies the following problem in $\mathbb{R}_+\times\mathbb{R}_+^d$:

$$
\begin{cases}
\partial_t\rho + \nabla_h\cdot(\rho\mathbf{u}_h) + \partial_y(\rho u_d) = 0,\\
\rho\{\partial_t\mathbf{u}_h + (\mathbf{u}_h\cdot\nabla_h + u_d\partial_y)\mathbf{u}_h\} + R\nabla_h(\rho\theta) = 0,\\
\partial_y(\rho\theta) = 0,\\
c_V\rho\{\partial_t\theta + (\mathbf{u}_h\cdot\nabla_h + u_d\partial_y)\theta\} + R\rho\theta(\nabla_h\cdot\mathbf{u}_h + \partial_y u_d) = \partial_y^2\theta,\\
u_d|_{y=0} = 0, \quad \lim\limits_{y\to+\infty}(\rho,\mathbf{u}_h,\theta) = (\rho^e,\mathbf{u}_h^e,\theta^e)(t,x',0),
\end{cases}
\tag{3.8}
$$

with the boundary values for θ :

$$
\begin{cases}
\partial_y\theta|_{y=0} = 0, & \text{when} \quad \alpha\neq 0,\\
\theta|_{y=0} = \theta^0(t,x'), & \text{when} \quad \alpha = 0,
\end{cases}
\tag{3.9}
$$

where $\theta^0(t,x') := \frac{\gamma(t,x')}{\beta(t,x')}$, provided $\beta\neq 0$.

Then, from the third equation and boundary conditions in (3.8), it implies that

$$
(\rho\theta)(t,x',y) \ \equiv \ (\rho^e\theta^e)(t,x',0) \ = \ \frac{p^e(t,x',0)}{R},
\tag{3.10}
$$

where p^e is the pressure of Euler flow and $p^e>0$. The relation (3.10) shows that there isn't boundary layer of size of $\mathcal{O}(\sqrt{\epsilon})$ for the pressure. Note that for the problem (3.11) endowed with the Neumann boundary condition for θ in (3.9), i.e., $\partial_y\theta|_{y=0}=0$, it is easy to check that

$$
(\rho,\mathbf{u}_h,\theta)(t,x',y) \ = \ (\rho^e,\mathbf{u}_h^e,\theta^e)(t,x',0), \quad u_d(t,x',y) = y\partial_{x_d}u_d^e(t,x',0)
$$

satisfies the problem (3.8). Indeed, we can investigate this by restricting the equations (3.4) on the boundary $\{x_d=0\}$ and using the boundary condition (3.5). In this case, it means that the state $(\rho,\mathbf{u},\theta)$ doesn't exist boundary layers of size of $\mathcal{O}(\sqrt{\epsilon})$. Therefore, we focus on the problem (3.8) with the Dirichlet boundary condition for θ in (3.9).

Plugging (3.10) into the problem (3.8)-(3.9), which can be reduced as the following problem in $\mathbb{R}_+ \times \mathbb{R}^d_+$ for the profile $(\mathbf{u}_h, u_d, \theta)(t, x', y)$:

$$
\begin{cases}
\partial_t \mathbf{u}_h + (\mathbf{u}_h \cdot \nabla_h + u_d \partial_y)\mathbf{u}_h + \frac{R\theta}{P}\nabla_h P = 0,\\
\partial_t \theta + (\mathbf{u}_h \cdot \nabla_h + u_d \partial_y)\theta = \frac{R}{(R+c_V)P}\theta\big(\partial_y^2\theta + \mathbf{u}_h \cdot \nabla_h P + P_t\big),\\
\nabla_h \cdot \mathbf{u}_h + \partial_y u_d = \frac{R}{(R+c_V)P}\partial_y^2\theta - \frac{c_V}{(R+c_V)P}\big(\mathbf{u}_h \cdot \nabla_h P + P_t\big),\\
(u_d, \theta)|_{y=0} = \big(0, \theta^0(t, x')\big), \quad \lim\limits_{y\to+\infty}(\mathbf{u}_h, \theta) = (\mathbf{U}_h, \Theta)(t, x)
\end{cases}
\tag{3.11}
$$

where the known functions

$$(P, \mathbf{U}_h, \Theta)(t, x') \ := \ (p^e, \mathbf{u}^e_h, \theta^e)(t, x', 0)$$

given by the Euler flow, and satisfy by using (3.4)-(3.5),

$$
\begin{cases}
\partial_t \mathbf{U}_h + \mathbf{U}_h \cdot \nabla_h \mathbf{U}_h + \frac{R\Theta}{P}\nabla_h P = 0,\\
\partial_t \Theta + \mathbf{U}_h \cdot \nabla_h \Theta - \frac{R\Theta}{(R+c_V)P} \cdot (P_t + \mathbf{U}_h \cdot \nabla_h P) = 0.
\end{cases}
\tag{3.12}
$$

We endow the problem (3.11) with the initial data

$$(\mathbf{u}_h, \theta)(0, x', y) \ = \ (\mathbf{u}_{h0}, \theta_0)(x', y), \tag{3.13}$$

and under the compatibility condition of $\mathbf{u}_{h0}$:

$$\lim_{y\to+\infty} \mathbf{u}_{h0} \ = \ \mathbf{U}_h(0, x'), \tag{3.14}$$

we can remove the infinity condition for $\mathbf{u}_h$ as $y \to +\infty$ in (3.11), since the condition $\lim\limits_{y\to+\infty} \mathbf{u}_h = \mathbf{U}_h(t, x')$ holds automatically from $(3.12)_1$ and (3.14). Therefore, we obtain the following initial-boundary value problem in $\mathbb{R}_+ \times \mathbb{R}^d_+$:

$$
\begin{cases}
\partial_t \mathbf{u}_h + (\mathbf{u}_h \cdot \nabla_h + u_d \partial_y)\mathbf{u}_h + \frac{R\theta}{P}\nabla_h P = 0,\\
\partial_t \theta + (\mathbf{u}_h \cdot \nabla_h + u_d \partial_y)\theta = \frac{\kappa\theta}{P}\big(\partial_y^2\theta + \mathbf{u}_h \cdot \nabla_h P + P_t\big),\\
\nabla_h \cdot \mathbf{u}_h + \partial_y u_d = \frac{\kappa}{P}\partial_y^2\theta - \frac{1-\kappa}{P}\big(\mathbf{u}_h \cdot \nabla_h P + P_t\big),\\
(u_d, \partial_y\theta)|_{y=0} = 0, \quad \lim\limits_{y\to+\infty}\theta(t, x, y) = \Theta(t, x'),\\
(\mathbf{u}_h, \theta)|_{t=0} = (\mathbf{u}_{h0}, \theta_0)(x', y),
\end{cases}
\tag{3.15}
$$

with the constant $\kappa := \frac{R}{R+c_V}$. In this paper, we focus on a simple case of the problem (3.15), i.e., the pressure $P(t, x')$ of the outflow is a positive function depending only on the variable t,

$$P(t, x') \ \equiv \ P(t) > 0.$$

Consequently, the problem (3.15) is reduced as follows:

$$
\begin{cases}
\partial_t \mathbf{u}_h + (\mathbf{u}_h \cdot \nabla_h + u_d \partial_y)\mathbf{u}_h = 0,\\
\partial_t \theta + (\mathbf{u}_h \cdot \nabla_h + u_d \partial_y)\theta = \frac{\kappa}{P}\theta\partial_y^2\theta + \frac{\kappa P_t}{P}\theta,\\
\nabla_h \cdot \mathbf{u}_h + \partial_y u_d = \frac{\kappa}{P}\partial_y^2\theta - \frac{(1-\kappa)P_t}{P},\\
(u_d, \theta)|_{y=0}\big(0, \theta^0(t, x')\big), \quad \lim\limits_{y\to+\infty}\theta(t, x, y) = \Theta(t, x'),\\
(\mathbf{u}_h, \theta)|_{t=0} = (\mathbf{u}_{h0}, \theta_0)(x', y).
\end{cases}
\tag{3.16}
$$

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